

AFOSR-TR-89-0574

2

AD-A210 862

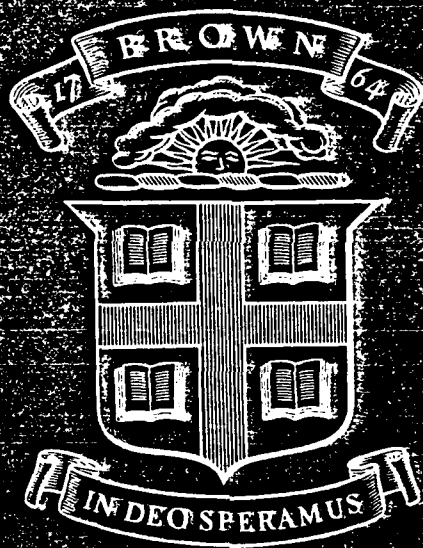
Wavefront Propagation for Reaction-Diffusion
Systems of PDE

by

G. Barles, L.C. Evans and P.E. Souganidis

March 1989

LCDS #89-3



Lefschetz Center for Dynamical Systems
and
Center for Control Sciences

89

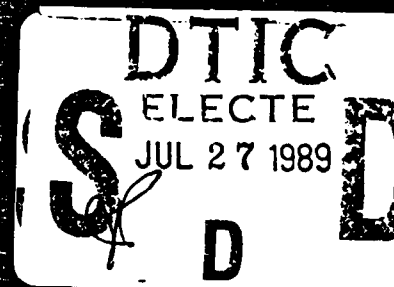
7

27

0110

DISTRIBUTION STATEMENT A

Approved for public release;
Distribution Unlimited



Division of Applied Mathematics

Brown University Providence RI 02912

①

Wavefront Propagation for Reaction-Diffusion
Systems of PDE

by

G. Barles, L.C. Evans and P.E. Souganidis

March 1989

LCDS #89-3

DTIC
ELECTR
JUL 27 1989

DISTRIBUTION STATEMENT A

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Wavefront Propagation for Reaction-Diffusion Systems of PDE		5. TYPE OF REPORT & PERIOD COVERED Preprint
7. AUTHOR(s) G.B. Ceremade, L.C. Evans, P.E. Souganidis		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Leifschetz Center for Dynamical Systems Division of Applied Mathematics Brown University, Providence, RI 02912		8. CONTRACT OR GRANT NUMBER(s) ISSA-860078
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research Bolling Air Force Base, Bldg 410 Washington, DC 20332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F, 2304, A1
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Same as 11		12. REPORT DATE March 1989
		13. NUMBER OF PAGES 35
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release: distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) N/A		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
20. ABSTRACT (We utilize the theory of viscosity solutions for Hamilton-Jacobi equations to study the asymptotic behavior of solutions to certain systems of reaction-diffusion PDE. Our principal result characterizes the region of convergence of the solution to an unstable rest point as the set where the solution of an appropriate Hamilton-Jacobi equation is positive.		

Wavefront Propagation for Reaction-Diffusion
Systems of PDE

by

G. Barles
CEREMADE
Université Paris IX-Dauphine
75775 Paris Cedex 16, France

L.C. Evans*
Department of Mathematics
University of Maryland
College Park, MD 20742 USA

P.E. Souganidis**
Division of Applied Mathematics
Brown University
Providence, RI 02912 USA

Abstract We utilize the theory of viscosity solutions for Hamilton-Jacobi equations to study the asymptotic behavior of solutions to certain systems of reaction-diffusion PDE. Our principal result characterizes the region of convergence of the solution to an unstable rest point as the set where the solution of an appropriate Hamilton-Jacobi equation is positive.

* Supported in part by the NSF under grant DMS-86-01532

** Supported in part by the NSF under grants DMS-8601258 and DMS-8657464, by the AFOSR under agreement AFOSR-ISSA-860078, by the ONR under contract N0014-83-K-0542 and by the ARO under contract DAAL03-86-K-0074.

Introduction

M.I. Freidlin ([9], [10]) has recently introduced probabilistic techniques into the study "wavefront propagation" for systems of reaction-diffusion PDE. The motivating idea is that should a reaction-diffusion system possess only a single unstable and a single stable equilibrium, then the solution u of the system will presumably tend to "switch" for large times from near the former to near the latter state. A mathematical problem is then to describe this transition, ideally in terms of simpler quantities than those governing the full, detailed behavior of the entire system of PDE. More precisely, to study the reaction-diffusion problem for large times of order ϵ^{-1} , Freidlin suggests a ϵ^{-1} rescaling in the space and time variables; so that our attention turns to the solutions u^ϵ of certain ϵ -dependent systems of PDE. We then hope to show that as $\epsilon \rightarrow 0$, the functions u^ϵ converge in some region $G \subset \mathbb{R}^n \times [0, \infty)$ to the stable point, and in the opposite region $[\mathbb{R}^n \times [0, \infty)] \setminus G$ to the unstable point. We simultaneously hope to describe geometrically or analytically this set G , whose boundary we envision as a spreading wavefront separating regions with quite different limiting behavior.

This paper, which is an extension to systems of earlier work [6] on single equations, brings to bear purely PDE techniques to this problem, especially the theory of viscosity solutions on Hamilton-Jacobi equations, due to Crandall-Lions [3]. The connection with the foregoing discussion is that, the region G alluded to above is the set where the solution J of a certain Hamilton-Jacobi equation is negative. Our procedure for understanding the limiting behavior of the solution of the reaction-diffusion system of PDE is thus first of all to build an appropriate Hamiltonian H out of the data given in the problem, second to solve the resulting Hamilton-Jacobi equation for J , and last to demonstrate the different limiting behavior of the solutions u^ϵ of the scaled system on the sets $\{J < 0\}$ and $\{J > 0\}$. We

informally regard the Hamiltonian as controlling somehow the rate of instability of the unstable point. We are thus able to characterize the asymptotic behavior of the "complicated" reaction-diffusion system in terms of the "simple" Hamilton-Jacobi equation. This possibility, first identified by Freidlin [9] in rather different terms, is attractive, but of course requires for this implementation many structural assumptions on the nonlinearities, which we list below. It would of course be quite interesting to extend our results, or at least the point of view espoused above, to systems with more general nonlinearities.

More precisely now, we intend to investigate the scaled reaction-diffusion system:

$$(1.1)_\varepsilon \quad \begin{cases} u_{k,t}^\varepsilon = \varepsilon d_k \Delta u_k^\varepsilon + \frac{1}{\varepsilon} f_k(u^\varepsilon) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u_k^\varepsilon = g_k & \text{on } \mathbb{R}^n \times \{0\} \quad (k=1, \dots, m). \end{cases}$$

Here the constants d_k ($1 \leq k \leq m$), and the functions

$$g: \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{and} \quad f: \mathbb{R}^m \rightarrow \mathbb{R}^m,$$

are given, where we write $g = (g_1, \dots, g_m)$, $f = (f_1, \dots, f_m)$. The unknown is $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$. We will assume that

$$d_k > 0 \quad (k=1, \dots, m),$$

and that the functions g, f are smooth, bounded and Lipschitz.

In addition we suppose that

$$(1.2) \quad g_k \geq 0 \quad (k=1, \dots, m)$$

and

$$(1.3) \quad G_0 = \{g_k > 0\} \quad (k=1, \dots, m)$$

is a bounded, smooth subset of \mathbb{R}^n . Under these assumptions there exists a unique smooth solution u^ε of the PDE (1.1) $_\varepsilon$, with

$$u_k^\varepsilon > 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \quad (k=1, \dots, m).$$

Our essential assumptions all concern the reaction term f . First of all we suppose

$$(F1) \quad f(0) = 0 ;$$

and also

$$(F2) \quad \begin{cases} f_k(u_1, \dots, u_{k-1}, 0, u_{k+1}, \dots, u_m) > 0 & \text{if} \\ u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_m \geq 0 \text{ and } u_l > 0 & \text{for some index } l \neq k. \end{cases}$$

Consequently the vector field f points strictly inward along the boundary of the positivity set

$$\Pi \equiv \{u \in \mathbb{R}^m \mid u_1 > 0, \dots, u_m > 0\},$$

except at the point 0 , which is an equilibrium point for the system (1.1) $_\varepsilon$.

To ensure that our solutions u^ε do not become unbounded as $\varepsilon \rightarrow 0$, we further hypothesize that

$$(F3) \quad \begin{cases} \text{there exists a constant } \Lambda \text{ such that} \\ f_k(u) \leq 0 & (k=1, \dots, m). \\ \text{if } u \in \Pi \text{ and } u_k \geq \Lambda. \end{cases}$$

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By <i>See Back Cover</i>	
Distribution/	
Availability Codes	
Avail and/or	
Dist	Special
A-1	

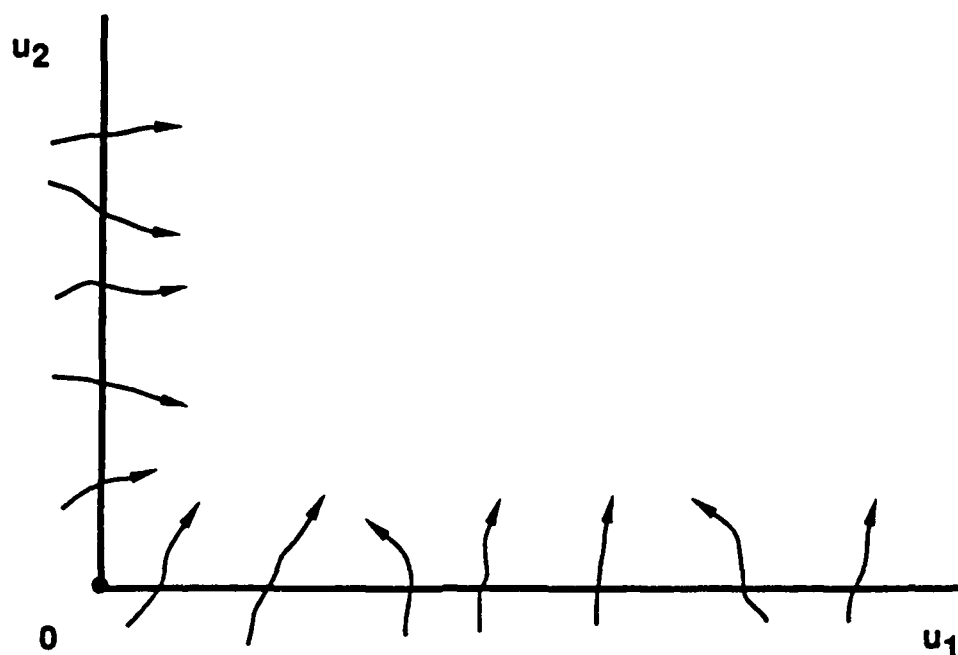


Figure 1

Next we set forth additional hypotheses which imply that the rest point 0 is unstable. Let us define the $m \times m$ matrix

$$C \equiv Df(0),$$

Df denoting the gradient of f . Thus

$$c_{kl} = f_{k, u_l}(0) \quad (1 \leq k, l \leq m).$$

We assume

$$(F4) \quad c_{kl} > 0 \quad (1 \leq k, l \leq m),$$

and

$$(F5) \quad f_k(u) \leq c_{kl} u_l \quad (u \in \Pi, k=1, \dots, m),$$

where we employ the standard summation convention.

for systems of the Kolmogorov-Petrovskii-Piskunov nonlinearity, discussed in [6].

Our main result, Theorem 1, asserts that under hypotheses (F1) - (F5) $u^\varepsilon(x, t)$ converges as $\varepsilon \rightarrow 0$ to zero or not depending on whether $J(x, t) > 0$ or $J(x, t) < 0$, the function J satisfying a Hamilton-Jacobi PDE whose structure we now describe. Given $p \in \mathbb{R}^n$, define the $m \times m$ matrix

$$B(p) \equiv \text{diag}(\dots, d_k |p|^2, \dots)$$

and then set

$$(1.4) \quad A(p) \equiv B(p) + C.$$

Now the matrix $A(p)$ has positive entries, and so Perron-Frobenius theory asserts that $A(p)$ possesses a simple, real eigenvalue $\lambda^0 = \lambda^0(A(p))$ satisfying

$$\text{Re } \lambda < \lambda^0$$

for all other eigenvalues λ of $A(p)$. Let us define then the Hamiltonian

$$(1.5) \quad H(p) \equiv \lambda^0(A(p)) \quad (p \in \mathbb{R}^n).$$

See, for instance, [7] for a review of the various properties of H , and in particular a proof that H is convex. We additionally set

$$(1.6) \quad L(q) \equiv \sup_{p \in \mathbb{R}^m} (p \cdot q - H(p));$$

L is the Lagrangian associated with H . Finally we define for each point $(x, t) \in \mathbb{R}^n \times (0, \infty)$ the action function

$$(1.7) \quad J(x, t) \equiv \inf \left\{ \int_0^t L(\dot{z}(s)) ds \mid z(0) \in G_0, z(t) = x \right\},$$

the infimum taken over all absolutely continuous functions

$$z: [0, t] \rightarrow \mathbb{R}^n$$

satisfying the stated initial and terminal conditions. As we will see, J turns out to be the (unique) solution of the Hamilton-Jacobi equation

$$(1.8) \quad \begin{cases} J_t + H(DJ) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ J = 0 & \text{on } G_0 \times \{0\} \\ J = +\infty & \text{on } \text{int}(\mathbb{R}^n - G_0) \times \{0\} \end{cases}$$

in the viscosity sense (cf. Crandall-Lions [3], Crandall-Evans-Lions [2], etc.).

Theorem 1. Under hypotheses (F1)-(F5) we have

$$(1.9) \quad \lim_{\epsilon \rightarrow 0} u^\epsilon = 0 \quad \text{uniformly on compact subset of } \{J > 0\}$$

and

$$(1.10) \quad \liminf_{\epsilon \rightarrow 0} u_k^\epsilon > 0 \quad (k=1, \dots, m) \text{ uniformly on compact subset of } \{J < 0\}.$$

We loosely interpret this theorem as describing how the Hamiltonian H , which depends upon both $C = Df(0)$ and the diffusion constants d_1, \dots, d_m , controls the instability of the equilibrium point $u = 0$.

Remark We should note also that it is possible to refine conclusion (1.10) by making further assumptions on the behavior of the vector field f in Π . As in Freidlin [9], we may for instance assume that there exists a unique equilibrium point $a \in \Pi$ which is asymptotically stable for the flow generated by the vector field f , as in the following illustration.

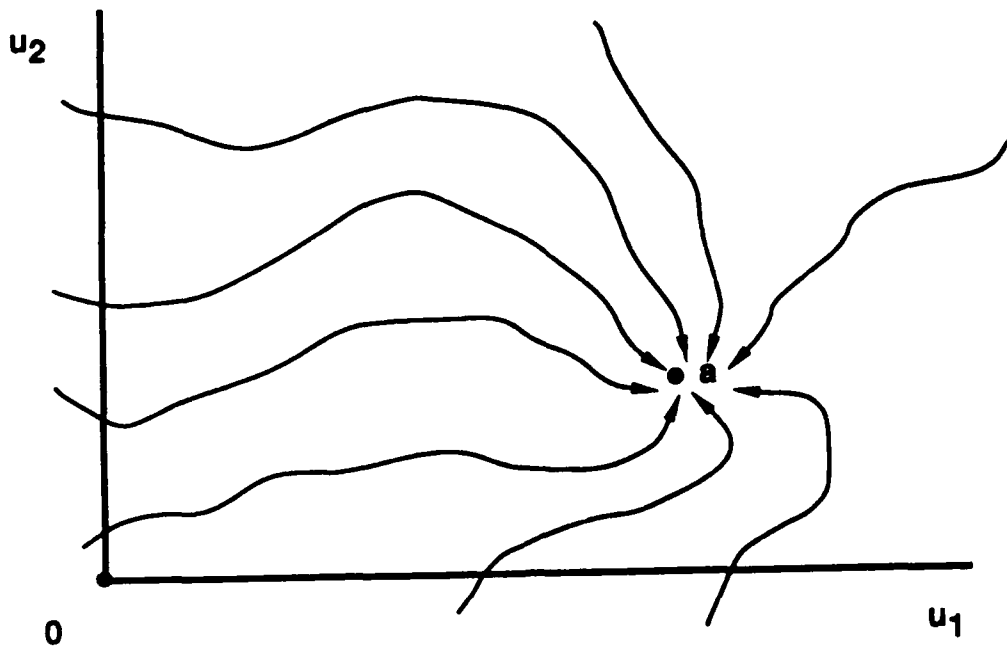


Figure 2

Under various fairly stringent technical assumptions, it is then possible to show that

$$(1.11) \quad \lim_{\epsilon \rightarrow 0} u^\epsilon = a \quad \text{uniformly on compact subsets of } \{J < 0\}.$$

We may consequently think of the boundary of the set $G \equiv \{J < 0\}$ as a "propagating wavefront" in the sense explained above.

Assertions (1.9), (1.11) are attractive, but the sufficient conditions we know to improve (1.10) to (1.11) are rather unsatisfactory technically. (For instance, we could assume that the diffusion constants are all equal and that f has a convex Lyapunov function associated with the stable point a , etc.). For this reason, we will not here formulate any precise assertions leading to (1.11), but instead refer the reader to Freidlin [9], [10] for more information on this point. \square

Our paper is organized so that the proof of Theorem 1 appears in §3, §2 being devoted to some preliminary estimates. The main idea, following [8],

is to perform a change of variables, after which we send $\varepsilon \rightarrow 0$. This approach is greatly complicated for the present case of systems, since we lose the maximum principle and consequently many of the estimates available in [6] for the scalar case. We depend instead upon some recent techniques introduced by Ishii [11] and Barles-Perthame [1] in Hamilton-Jacobi theory, which allow us to proceed with only supremum-norm bounds. This is a great advantage since better estimates seem to be unobtainable, but the price is a far greater intricacy in some of the convergence proofs. We will encounter for instance troubles in interpreting in just what sense certain solutions take on their initial values. An appendix (§4) develops some useful theory on such questions, which we will need in §3.

Finally we let us note that although Freidlin's work has greatly inspired us, we believe that the PDE techniques developed here (and in [6]) provide information which is not at all clear from the probabilistic viewpoint. We for instance do not require the fairly specific structural assumptions on the nonlinearity f utilized in [9, p.467].

2. Preliminary estimates

Henceforth we always suppose hypotheses (F1)-(F5) to be in effect.

Lemma 2.1 There exists a constant C_1 such that

$$(2.1) \quad 0 < u_k^\varepsilon \leq C_1 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

for $k=1, \dots, m$ and each $\varepsilon > 0$.

Proof. Choose a smooth, bounded, Lipschitz vector field

$$\hat{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m$$

such that

$$(2.2) \quad \begin{cases} \hat{f}(u) = f(u) & (u \in \bar{\Pi}) \\ \text{and} \\ \hat{f}_k(u) \geq 0 & \text{if } u_k < 0, k=1, \dots, m; \end{cases}$$

this is possible in light of (F2). Now let \hat{u}^ε be the unique, smooth solution of the system

$$(2.3)_\varepsilon \quad \begin{cases} \hat{u}_{k,t}^\varepsilon = \varepsilon d_k \Delta \hat{u}_k^\varepsilon + \frac{1}{\varepsilon} \hat{f}_k(\hat{u}^\varepsilon) & \text{in } \mathbb{R}^n \times (0, \infty) \\ \hat{u}_k^\varepsilon = g_k & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Choose

$$\eta: \mathbb{R} \rightarrow \mathbb{R}$$

to be smooth and convex, with

$$(2.4) \quad \begin{cases} \eta = 0 & \text{on } [0, \infty) \\ \eta > 0 & \text{on } (-\infty, 0) \end{cases}$$

Then for each $k=1, \dots, m$,

*Page 11 blank
misnumbered*

$$\frac{d}{dt} \int_{\mathbb{R}^n} \eta(u_k^\varepsilon) dt = - \varepsilon \int_{\mathbb{R}^n} d_k \eta''(u_k^\varepsilon) |Du_k^\varepsilon|^2 dx$$

$$= \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \eta'(u_k^\varepsilon) f_k(u^\varepsilon) dx$$

$$\leq 0, \quad \text{by (F3).}$$

Since

$$\int_{\mathbb{R}^n} \eta(g_k) dx = 0,$$

we discover

$$u_k^\varepsilon \leq \theta \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

3. Observe now that hypotheses (F1), (F2) and (F4) imply $f_k(u) \geq 0$ if $u_k^\varepsilon \geq 0$ is small enough ($k=1, \dots, m$). Thus the strong maximum principle for parabolic equations implies

$$u_k^\varepsilon > 0 \quad \text{in } \mathbb{R}^n \times [0, \infty), \quad (k=1, \dots, m) \quad \square$$

We next introduce, following [5], [6], [7], [8], etc. the new functions

$$(2.5) \quad v_k^\varepsilon = - \varepsilon \log u_k^\varepsilon \quad \text{in } \mathbb{R}^n \times (0, \infty) \quad (k=1, \dots, m).$$

A calculation shows that $v^\varepsilon = (v_1^\varepsilon, \dots, v_m^\varepsilon)$ satisfies the system

$$(2.6)_\varepsilon \quad \begin{cases} v_{k,t}^\varepsilon - \varepsilon d_k \Delta v_k^\varepsilon + d_k |Dv_k^\varepsilon|^2 = - \frac{f_k(u^\varepsilon)}{u_k^\varepsilon} & \text{in } \mathbb{R}^n \times (0, \infty) \\ v_k^\varepsilon = - \varepsilon \log g_k & \text{on } G_0 \times \{0\} \\ v_k^\varepsilon = + \infty & \text{on } \text{int}(\mathbb{R}^n - G_0) \times \{0\}. \end{cases}$$

Note also that in view of Lemma 2.1 we have

$$(2.7) \quad v_k^\varepsilon \geq - \varepsilon \log C_1 \quad \text{in } \mathbb{R}^n \times [0, \infty)$$

for $k=1, \dots, m$ and each $\varepsilon > 0$.

Lemma 2.2 For each compact subset $Q \subset \subset (\mathbb{R}^n \times (0, \infty)) \cup (G_0 \times [0, \infty))$ there exists a constant $C_2(Q)$ such that

$$|v_k^\varepsilon| \leq C_2(Q) \quad \text{in } Q$$

for $k = 1, \dots, m$ and each $\varepsilon > 0$.

Proof 1. Let O denote any open subset of $\mathbb{R}^n \times (0, \infty)$. Suppose that ϕ is a smooth function satisfying

$$(2.8) \quad \phi_t - \varepsilon d_k \Delta \phi + d_k |D\phi|^2 \geq \lambda \quad \text{in } O,$$

where $\lambda > 0$ is a positive constant to be selected later. Assume now that $v_k^\varepsilon - \phi$ has a maximum at some point $(x_0, t_0) \in O$. Then at the point (x_0, t_0) we have

$$\begin{aligned} 0 &\leq \left[u_k^\varepsilon - \phi \right]_t - \varepsilon d_k \Delta \left[u_k^\varepsilon - \phi \right] \\ &\leq - \frac{f_k(u^\varepsilon)}{u_k^\varepsilon} - \lambda \quad \text{by (2.6)}_\varepsilon \text{ and (2.8).} \end{aligned}$$

But

$$\begin{aligned} -f_k(u^\varepsilon) &= -f_k(\dots, u_{k-1}^\varepsilon, 0, u_{k+1}^\varepsilon, \dots) \\ &+ \left[f_k(\dots, u_{k-1}^\varepsilon, 0, u_{k+1}^\varepsilon, \dots) - f_k(\dots, u_{k-1}^\varepsilon, u_k^\varepsilon, u_{k+1}^\varepsilon, \dots) \right] \\ &\leq C u_k^\varepsilon, \quad \text{according to Lemma 2.1 and (F2).} \end{aligned}$$

Substituting above we discover

$$0 \leq C - \lambda,$$

a contradiction for λ large enough.

2. We may assume that for some $r > 0$ the ball $B(0, r)$ lies in G_0 . Set then

$$\phi^1 = \frac{1}{r^2 - |x|^2} + \beta t + \gamma \quad (|x| < r, t \geq 0)$$

β and γ picked as indicated below (cf. [5], [7]). We compute

$$\begin{aligned} \phi_t^1 &= \epsilon d_k \Delta \phi^1 + d_k |D\phi^1|^2 \\ &= \beta - \epsilon d_k \left(\frac{2n}{r^2 - |x|^2} + \frac{8|x|^2}{(r^2 - |x|^2)^3} \right) + d_k \frac{4|x|^2}{(r^2 - |x|^2)^4} \\ &\geq \lambda \quad \text{in } B(0, r) \times (0, \infty) \quad (k=1, \dots, m), \end{aligned}$$

provided $\beta > 0$ is large enough. If we then adjust γ to be so large that

$$\phi^1 \geq v_k^\epsilon \quad \text{on } B(0, r) \times \{0\},$$

we deduce from Step 1 that

$$v_k^\epsilon \leq \phi^1 \quad \text{in } B(0, r) \times (0, \infty).$$

Consequently

$$(2.9) \quad v_k^\epsilon \leq C \quad \text{in } B(0, r/2) \times (0, T).$$

for $k=1, \dots, m$ and each $\epsilon > 0$, $T > 0$.

3. Now write

$$\phi^2 = \frac{\alpha|x|^2}{t} + \beta t + \gamma \quad (|x| > \frac{r}{2}, t > 0).$$

We calculate

$$\phi_t^2 - \epsilon d_k \Delta \phi^2 + d_k |D\phi^2|^2 = -\frac{\alpha|x|^2}{t} + \beta - \epsilon d_k \frac{2n}{t} + d_k \frac{4\alpha^2|x|^2}{t^2}$$

$$\geq \lambda \quad \text{in } (\mathbb{R}^n - B(0, r/2)) \times (0, \infty),$$

provided $\alpha, \beta > 0$ and large enough. We now pick γ large enough to ensure

$$\phi^2 \geq v_k^\varepsilon \quad \text{on } \partial B(0, r/2) \times (0, T).$$

This is possible owing to estimate (2.9). Consequently Step 1 implies

$$v_k^\varepsilon \leq \phi^2 \quad \text{in } (\mathbb{R}^n - B(0, r/2)) \times (0, T)$$

for $k=1, \dots, m$ and each $\varepsilon > 0, T > 0$.

This bound and (2.9) lead at once to the estimate stated in the Lemma. \square

3. Proof of Theorem 1.

For each $(x, t) \in \mathbb{R}^n \times (0, \infty)$ we set

$$(3.1) \quad v_{k*}(x, t) \equiv \liminf_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}} v_k^\varepsilon(y, s) \quad (k=1, \dots, m)$$

and

$$(3.2) \quad v_k^*(x, t) \equiv \limsup_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}} v_k^\varepsilon(y, s) \quad (k=1, \dots, m).$$

We additionally write

$$(3.3) \quad v_* \equiv \min_{1 \leq k \leq m} v_{k*}, \quad v^* \equiv \max_{1 \leq k \leq m} v_k^*.$$

We intend to show that v_* is a lower semicontinuous supersolution, and v^* is a upper semicontinuous subsolution, of an appropriate variational inequality involving the Hamiltonian H defined in §1.

Proposition 3.1. We have

(i)

$$(3.4) \quad \min(v_{*t} + H(Dv_*), v_*) \geq 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

in the viscosity sense, and

(ii)

$$(3.5) \quad v_* = \begin{cases} 0 & \text{on } G_0 \times \{0\} \\ +\infty & \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\}. \end{cases}$$

Proof 1. Because of (2.7) it is clear that

$$v_* \geq 0.$$

To demonstrate that

$$v_{*t} + H(Dv_*) \geq 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

in the viscosity sense, we fix a smooth test function ϕ and assume

$$(3.6) \quad \begin{cases} v_* - \phi \text{ has a strict local minimum at some point} \\ (x_0, t_0) \in \mathbb{R}^n \times (0, \infty). \end{cases}$$

We must prove

$$(3.7) \quad \phi_t + H(D\phi) \geq 0 \quad \text{at} \quad (x_0, t_0).$$

2. Let (ψ_1, \dots, ψ_k) be a positive eigenvector of the matrix $A(D\phi(x_0, t_0))$ corresponding to the principal eigenvalue $\lambda^0(A(D\phi(x_0, t_0))) = H(D\phi(x_0, t_0))$.

Note then

$$(3.8) \quad v_*(x, t) = \liminf_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}} \min_{1 \leq l \leq m} \left[v_1^\varepsilon(y, s) + \varepsilon \log \psi_l \right].$$

Combining (3.6) and (3.8), we deduce that there exists an index $k \in \{1, \dots, m\}$, a sequence $\varepsilon_r \rightarrow 0$, and points $(x^{\varepsilon_r}, t^{\varepsilon_r})$ such that

$$(3.9) \quad v_k^{\varepsilon_r} \left[x^{\varepsilon_r}, t^{\varepsilon_r} \right] + \varepsilon_r \log \psi_k = \min_{1 \leq l \leq m} \left[v_l^{\varepsilon_r} \left[x^{\varepsilon_r}, t^{\varepsilon_r} \right] + \varepsilon_r \log \psi_l \right] \rightarrow v_*(x_0, t_0),$$

$$(3.10) \quad \left[v_k^{\varepsilon_r} + \varepsilon_r \log \psi_k \right] - \phi \text{ has a local minimum at } \left[x^{\varepsilon_r}, t^{\varepsilon_r} \right],$$

and

$$(3.11) \quad \left[x^{\varepsilon_r}, t^{\varepsilon_r} \right] \rightarrow (x_0, t_0) \quad \text{as} \quad r \rightarrow \infty.$$

Applying then the maximum principle, we obtain from (2.6)_c the estimate

$$(3.12) \quad \begin{aligned} 0 &\leq \phi_t - \varepsilon_r d_k \Delta \phi + d_k |D\phi|^2 + \frac{f_k(u_k^{\varepsilon_r})}{u_k^{\varepsilon_r}} \\ &\leq \phi_t - \varepsilon_r d_k \Delta \phi + d_k |D\phi|^2 + c_{kl} \exp \frac{v_k^{\varepsilon_r} - v_l^{\varepsilon_r}}{\varepsilon_r} \end{aligned}$$

at the point $[x^{\varepsilon_r}, t^{\varepsilon_r}]$, the second inequality being a consequence of hypothesis (F5). Using (3.9) and (3.11), we simplify (3.12) to read

$$0 \leq \phi_t + d_k |D\phi|^2 + c_{kl} \frac{\psi_1}{\psi_k} + o(1) \quad \text{as } \varepsilon_r \rightarrow 0$$

at (x_0, t_0) . Since

$$d_k |D\phi|^2 \psi_k + c_{kl} \psi_1 = (A(D\phi(x_0, t_0))\psi)_k = H(D\phi(x_0, t_0))\psi_k,$$

we deduce (3.7) upon letting $\varepsilon_r \rightarrow 0$.

3. We next verify assertion (ii). To do so, we fix $\mu > 0$ and select $\zeta \in C^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} \zeta = 0 & \text{on } \bar{G}_0, \zeta > 0 & \text{on } \mathbb{R}^n - \bar{G}_0 \\ 0 \leq \zeta \leq 1. \end{cases}$$

We now claim that

$$(3.13) \quad \max(v_{*t} + H(Dv_*), v_* - \mu\zeta) \geq 0 \quad \text{on } \mathbb{R}^n \times \{0\}$$

in the viscosity sense, which means that if ϕ is a smooth test function and

$$(3.14) \quad \begin{cases} v_* - \phi \text{ has a strict local minimum} \\ \text{at some point } (x_0, 0) \in \mathbb{R}^n \times \{0\}, \end{cases}$$

then either

$$(3.15) \quad v_*(x_0, 0) \geq \mu\zeta(x_0)$$

or else

$$(3.16) \quad \phi_t + H(D\phi) \geq 0 \quad \text{at } (x_0, 0).$$

Now if $x_0 \in \bar{G}_0$, then (3.15) is clearly true. Otherwise suppose $x_0 \in \mathbb{R}^n - \bar{G}_0$ and

$$v_*(x_0, 0) < \mu\zeta(x_0) < \infty.$$

We repeat now the argument from Steps 1-2, noting in particular that since

$$v_k^{\varepsilon_r}(x, 0) = +\infty \quad \text{for all } x \text{ near } x_0,$$

the points $\{x^{\varepsilon_r}, t^{\varepsilon_r}\}$ above lie in $\mathbb{R}^n \times (0, \infty)$. As such the maximum principle argument leading to inequality (3.12) is valid, and the rest of the proof proceeds as before, yielding at last the inequality (3.16).

4. Since

$$v_k^{\varepsilon}(x, 0) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (k=1, \dots, m)$$

if $x_0 \in G_0$, we have

$$v_* = 0 \quad \text{on } G_0 \times \{0\}.$$

To see that

$$(3.22) \quad v_* = +\infty \quad \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\},$$

choose any point $x_0 \in \mathbb{R}^n - \bar{G}_0$ and suppose instead

$$(3.17) \quad v_*(x, 0) < \infty.$$

Fix $\delta > 0$ and then define

$$\phi^\delta(x, t) = -\frac{|x-x_0|^2}{\delta} - \lambda t,$$

for $\lambda = \lambda(\delta)$ to be selected below. Since v_* is lower semicontinuous,

(3.18) $v_* - \phi^\delta$ has a minimum at a point $(x_\delta, t_\delta) \in \mathbb{R}^n \times [0, \infty)$.

Then

$$(3.19) \quad \frac{|x_\delta - x_0|^2}{\delta} \leq v_*(x_\delta, t_\delta) + \frac{|x_\delta - x_0|^2}{\delta} + \lambda t_\delta \leq v_*(x_0, 0) < \infty.$$

Now if $t_\delta > 0$,

$$\phi_t^\delta + H(D\phi^\delta) \geq 0 \quad \text{at } (x_\delta, t_\delta);$$

whence

$$(3.20) \quad -\lambda + H\left[-\frac{2(x_\delta - x_0)}{\delta}\right] \geq 0,$$

a contradiction for $\lambda = \lambda(\delta)$ sufficiently large. Thus $t_\delta = 0$. If

$$v_*(x_0, 0) < \mu \zeta(x_0),$$

then (3.19) implies

$$v_*(x_\delta, 0) < \mu \zeta(x_\delta)$$

for small enough δ , and so according to (3.13) we once more would obtain

(3.20). Thus

$$v_*(x_0, 0) \geq \mu \zeta(x_0).$$

But since $\zeta(x_0) > 0$ and $\mu > 0$ is arbitrary, (3.17) cannot be true. \square

Following next is the analogue of Proposition 3.1, with v^* in place of v_* .

Proposition 3.2. We have

(1)

$$(3.21) \quad \min (v_t^* + H(Dv^*), v^*) \leq 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

in the viscosity sense, and

(11)

$$(3.22) \quad v^* = \begin{cases} 0 & \text{on } G_0 \times \{0\} \\ +\infty & \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\}. \end{cases}$$

Proof 1. Since $v^* \geq 0$, to establish (3.21) we must show

$$v_t^* + H(Dv^*) \leq 0 \quad \text{on the set } \{v^* > 0\},$$

in the viscosity sense. So select any smooth test function ϕ and suppose

$$(3.23) \quad \begin{cases} v^* - \phi \text{ has a strict local maximum at some point} \\ (x_0, t_0) \in \mathbb{R}^n \times (0, \infty), \end{cases}$$

with

$$(3.24) \quad v^*(x_0, t_0) > 0.$$

We need to show

$$(3.25) \quad \phi_t + H(D\phi) \leq 0 \quad \text{at } (x_0, t_0).$$

Let (ψ_1, \dots, ψ_k) be a positive eigenvector of $A(D\phi(x_0, t_0))$, corresponding to the principal eigenvalue $\lambda^0(A(D\phi(x_0, t_0)) = H(D\phi(x_0, t_0)))$. Then

$$(3.26) \quad v^*(x, t) = \limsup_{\substack{\varepsilon \rightarrow 0 \\ (y, s) \rightarrow (x, t)}} \max_{1 \leq l \leq m} \left[v_1^\varepsilon(y, s) + \varepsilon \log \psi_l \right].$$

Combining (3.23) and (3.26), we deduce that there exists an index

$k \in \{1, \dots, m\}$, a sequence $\varepsilon_r \rightarrow 0$, and points $(x^{\varepsilon_r}, t^{\varepsilon_r})$ such that

$$(3.27) \quad v_k^{\varepsilon_r}(x^{\varepsilon_r}, t^{\varepsilon_r}) + \varepsilon_r \log \psi_k = \max_{1 \leq l \leq m} \left[v_l^{\varepsilon_r}(x^{\varepsilon_r}, t^{\varepsilon_r}) + \varepsilon_r \log \psi_l \right] \rightarrow v^*(x_0, t_0),$$

$$(3.28) \quad v_k^{\varepsilon_r}(x^{\varepsilon_r}, t^{\varepsilon_r}) + \varepsilon_r \log \psi_k - \phi \text{ has a local maximum at } (x^{\varepsilon_r}, t^{\varepsilon_r}).$$

and

$$(3.29) \quad \left(x^{\varepsilon_r}, t^{\varepsilon_r} \right) \rightarrow (x_0, t_0) \quad \text{as } r \rightarrow \infty.$$

Utilizing the maximum principle, we deduce from (2.6)_ε the inequality

$$(3.30) \quad \phi_t - \varepsilon_r d_k \Delta \phi + d_k |D\phi|^\varepsilon \leq - \frac{f_k(u^{\varepsilon_r})}{u_k^{\varepsilon_r}}$$

at the point $\left(x^{\varepsilon_r}, t^{\varepsilon_r} \right)$. We must now study the limiting behavior of the term on the right hand side as $r \rightarrow \infty$.

2. We assert

$$(3.31) \quad u_l^{\varepsilon_r} \left(x^{\varepsilon_r}, t^{\varepsilon_r} \right) \rightarrow 0 \quad (l = 1, \dots, m)$$

as $r \rightarrow \infty$. To see this, note that

$$u_k^{\varepsilon_r} \left(x^{\varepsilon_r}, t^{\varepsilon_r} \right) = \exp \left(- \frac{v_k^{\varepsilon_r} \left(x^{\varepsilon_r}, t^{\varepsilon_r} \right)}{\varepsilon_r} \right) \rightarrow 0,$$

owing to (3.24) and (3.27). This establishes (3.31) for $l = k$. Now suppose additionally that for some index $l \neq k$,

$$\limsup_{r \rightarrow \infty} u_l^{\varepsilon_r} \left(x^{\varepsilon_r}, t^{\varepsilon_r} \right) = a > 0.$$

Then passing if necessary through an appropriate subsequence

$$\left\{ \varepsilon_s \right\}_{s=1}^{\infty} \subset \left\{ \varepsilon_r \right\}_{r=1}^{\infty}, \quad \text{we find}$$

$$f_k \left(u^{\varepsilon_s} \left(x^{\varepsilon_s}, t^{\varepsilon_s} \right) \right) \rightarrow f_k(\dots, a, \dots, 0, \dots),$$

$a > 0$ occupying the l^{th} argument and 0 occupying the k^{th} argument of f_k .

Using hypothesis (F2), we obtain

$$\lim_{s \rightarrow \infty} \frac{f_k \left[u^{\varepsilon_s} \left[x^{\varepsilon_s}, t^{\varepsilon_s} \right] \right]}{u_k^{\varepsilon_s} \left[x^{\varepsilon_s}, t^{\varepsilon_s} \right]} = +\infty,$$

a conclusion at variance with (3.30). Thus statement (3.31) is true.

And indeed the exact reasoning above proves additionally

$$(3.32) \quad \frac{f_k \left(u^{\varepsilon_r} \right)}{u_k^{\varepsilon_r}} = o(1)$$

as $r \rightarrow \infty$, the functions evaluated at the point $\left[x^{\varepsilon_r}, t^{\varepsilon_r} \right]$. We further refine this observation by claiming

$$(3.33) \quad \frac{f_k \left(u^{\varepsilon_r} \right)}{u_k^{\varepsilon_r}} = \frac{c_{k1} u_1^{\varepsilon_r}}{u_k^{\varepsilon_r}} + o(1)$$

as $r \rightarrow \infty$. To see this we observe that

$$(3.34) \quad \begin{aligned} f_k(u) &= f_k(0) + c_{k1} u_1 + o(|u|^2) \\ &= c_{k1} u_1 + o(|u|^2) \quad (u \in \mathbb{R}^m). \end{aligned}$$

Now (3.31) yields

$$\frac{|u^{\varepsilon_r}|^2}{u_k^{\varepsilon_r}} = o(1) \sum_{l=1}^m \frac{u_l^{\varepsilon_r}}{u_k^{\varepsilon_r}} \leq o(1) \frac{c_{k1} u_1^{\varepsilon_r}}{u_k^{\varepsilon_r}}$$

as $r \rightarrow \infty$. Consequently (3.34) implies

$$\frac{f_k \left(u^{\varepsilon_r} \right)}{u_k^{\varepsilon_r}} \geq (1-o(1)) \frac{c_{k1} u_1^{\varepsilon_r}}{u_k^{\varepsilon_r}}$$

as $r \rightarrow \infty$. In view of (3.32) then

$$(3.35) \quad \frac{c_{kl} u_l^{\varepsilon_r}}{u_k^{\varepsilon_r}} = o(1)$$

as $r \rightarrow \infty$. Since

$$\frac{f_k(u^{\varepsilon_r})}{u_k^{\varepsilon_r}} \leq \frac{c_{kl} u_l^{\varepsilon_r}}{u_k^{\varepsilon_r}}$$

according to hypothesis (F5), claim (3.33) is proved.

Finally note that (3.35) implies

$$(3.45) \quad \frac{u_l^{\varepsilon_r}}{u_k^{\varepsilon_r}} = o(1) \quad (l=1, \dots, m)$$

as $r \rightarrow \infty$.

3. Owing to (3.27) and (3.33), inequality (3.30) yields

$$\phi_t + d_k |D\phi|^{\varepsilon} + c_{kl} \frac{\psi_1}{\psi_k} \leq o(1) \quad \text{as } \varepsilon_r \rightarrow \infty$$

at the point (x_0, t_0) . We now conclude as in the proof of Proposition 3.1(1)

4. Next we verify assertion (ii). We first claim

$$(3.37) \quad \min (v_t^* + H(Dv^*), v^*) \leq 0 \quad \text{on } G_0 \times \{0\}$$

in the viscosity sense, which means that if ϕ is a smooth test function and

$$\begin{cases} v^* - \phi \text{ has a strict local maximum at some point} \\ (x_0, 0) \in G_0 \times \{0\} \end{cases}$$

then either

$$(3.38) \quad v^*(x_0, 0) = 0$$

or else

$$(3.39) \quad \phi_t + H(D\phi) \leq 0 \quad \text{at } (x_0, 0).$$

Now if (3.38) is false, then we repeat the argument from Steps 1-3, above noting that since

$$v_k^\varepsilon(x, 0) = 0 \quad \text{for all } x \text{ near } x_0,$$

the points $(x^{\varepsilon_r}, t^{\varepsilon_r})$ lie in $\mathbb{R}^n \times (0, \infty)$. Consequently the maximum principle arguments employed above lead us to (3.39).

5. Now observe that

$$v^* = +\infty \quad \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\},$$

since $v_k^\varepsilon = +\infty$ on that set. Suppose then that $x_0 \in G_0$, but

$$(3.40) \quad v^*(x_0, 0) > 0.$$

Fix $\delta > 0$ and write

$$\phi^\delta \equiv \frac{|x - x_0|^2}{\delta} + \lambda t$$

for $\lambda = \lambda(\delta)$ to be chosen. Since v^* is upper semicontinuous and is bounded near x_0 ,

$$(3.41) \quad \begin{cases} v^* - \phi^\delta \text{ has a local maximum at a point} \\ (x_\delta, t_\delta) \in G_0 \times [0, \infty) \end{cases}$$

for each sufficiently small $\delta > 0$, with

$$x_\delta \rightarrow x_0 \quad \text{as } \delta \rightarrow 0.$$

If $t_\delta > 0$, then

$$\phi_t^\delta + H(D\phi^\delta) \leq 0 \quad \text{at } (x_\delta, t_\delta);$$

whence

$$(3.42) \quad \lambda + H\left[\frac{2(x_\delta - x_0)}{\delta}\right] \leq 0,$$

a contradiction for $\lambda = \lambda(\delta)$ large enough. Thus $t_\delta = 0$.

Now

$$0 < v^* \left[x_0, 0 \right] \leq v^* \left[x_\delta, 0 \right] - \frac{|x_\delta - x_0|^2}{\delta};$$

owing to (3.37) we once again reach the contradiction (3.42). Hence (3.41) is untenable, so that

$$v^* = 0 \quad \text{on } G_0,$$

as required. \square

Conclusion of the proof of Theorem 1.

In light of Propositions 3.1 and 3.2 we may invoke the uniqueness theorems developed in the Appendix, §4, to find

$$v^* = v_* = I \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

where I is the unique viscosity solution of the Hamilton-Jacobi variational inequality

$$(3.43) \quad \begin{cases} \min(I_t + H(DI), I) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ I = 0 & \text{on } G_0 \times \{0\} \\ I = +\infty & \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\} . \end{cases}$$

Additionally,

$$(3.44) \quad v_k^\varepsilon \rightarrow I \quad \text{uniformly on compact subsets of } \mathbb{R}^n \times (0, \infty), \quad k=1, \dots, m.$$

Now according to §5 in [6], we have

$$(3.45) \quad I = \max(J, 0),$$

where J is the unique viscosity solution of the Hamilton-Jacobi equation

$$(3.46) \quad \begin{cases} J_t + H(DJ) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ J = 0 & \text{on } G_0 \times \{0\} \\ J = +\infty & \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\} \end{cases}$$

In addition we have the representation formula

$$(3.47) \quad J(x, t) = \inf \left\{ \int_0^t L(\dot{z}(s)) ds \mid z(0) \in G_0, z(t) = x \right\},$$

the infimum taken over all absolutely continuous functions

$$z: [0, t] \rightarrow \mathbb{R}^n.$$

Now since (2.5) and (3.44) imply

$$u_k^\varepsilon = e^{-\frac{v_k^\varepsilon}{\varepsilon}} = e^{-\frac{I+o(1)}{\varepsilon}} \quad (k=1, \dots, m),$$

we see that

$$(3.48) \quad u_k^\varepsilon \rightarrow 0 \quad \text{uniformly on compact subsets of } \{I > 0\} = \{J > 0\}$$

as $\varepsilon \rightarrow 0$, for $k=1, \dots, m$.

We must now show

$$(3.49) \quad \liminf_{\varepsilon \rightarrow 0} \min_{1 \leq k \leq m} u_k^\varepsilon > 0 \quad \text{uniformly on compact subset of } \{J < 0\}.$$

So fix any point $(x_0, t_0) \in \{J < 0\}$. Then $I = 0$ near (x_0, t_0) . Define

$$\phi(x, t) = |x - x_0|^2 + |t - t_0|^2$$

Owing to (3.44) we see that for each $k=1, \dots, m$.

$$(3.50) \quad v_k^\varepsilon - \phi \quad \text{has a maximum at a point } (x_k^\varepsilon, t_k^\varepsilon),$$

with

$$(3.51) \quad (x_k^\varepsilon, t_k^\varepsilon) \rightarrow (x_0, t_0) \quad \text{as } \varepsilon \rightarrow 0.$$

Applying the maximum principle, we find using (2.6)_ε that

$$(3.52) \quad o(1) = \phi_t - \varepsilon d_k \Delta \phi + d_k |D\phi|^2 \leq - \frac{f_k(u^\varepsilon)}{u_k^\varepsilon};$$

and so

$$(3.53) \quad f_k(u^\varepsilon) \leq o(1) u_k^\varepsilon \quad \text{at the point } (x_k^\varepsilon, t_k^\varepsilon),$$

as $\varepsilon \rightarrow 0$.

Now there exists a constant $\alpha > 0$ such that

$$(3.54) \quad f^k(u) \geq c_{k1} u_1 - \alpha |u|^2 \quad \text{for all } u \in \mathbb{R}^m.$$

Let us suppose first that

$$(3.55) \quad u_l^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) \leq \frac{c_{k1}}{2\alpha} \quad \text{for each } l=1, \dots, m, l \neq k.$$

Then from (3.53) and (3.54) we deduce

$$o(1)u_k^\varepsilon \geq f_k(u^\varepsilon) \geq \sum_{l \neq k} c_{kl} u_l^\varepsilon + c_{kk} u_k^\varepsilon - \alpha |u^\varepsilon|^2$$

at $(x_k^\varepsilon, t_k^\varepsilon)$; whence (3.55) implies

$$(3.56) \quad u_k^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) \geq \frac{c_{kk}}{2\alpha}$$

for ε small enough. Should (3.55) fail, then

$$u_l^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) > \frac{c_{k1}}{2\alpha} \quad \text{for some } l \neq k.$$

But then owing to hypothesis (F2)

$$f^k(\dots u_{k-1}^\varepsilon, 0, u_{k+1}^\varepsilon, \dots) \geq \beta$$

at the point $(x_k^\varepsilon, t_k^\varepsilon)$, for some positive constant β . Thus at $(x_k^\varepsilon, t_k^\varepsilon)$ we

have

$$f^k(u^\varepsilon) \geq \beta + f^k(u^\varepsilon) - f^k(\dots u_{k-1}^\varepsilon, 0, u_{k+1}^\varepsilon, \dots)$$

$$\geq \beta + c_{kk} u_k^\varepsilon - \gamma \left[(u_k^\varepsilon)^2 + \sum_{l=1}^m u_k^\varepsilon u_l^\varepsilon \right]$$

for some $\gamma > 0$. Consequently (3.53) implies

$$(u_k^\varepsilon)^2 \geq \beta + o(|u_k^\varepsilon|) \quad \text{at} \quad (x_k^\varepsilon, t_k^\varepsilon),$$

and so

$$u_k^\varepsilon (x_k^\varepsilon, t_k^\varepsilon) \geq \delta > 0$$

for sufficiently small $\delta > 0$. But since

$$(v_k^\varepsilon - \phi)(x_k^\varepsilon, t_k^\varepsilon) \geq (v_k^\varepsilon - \phi)(x_0, t_0),$$

we have

$$v_k^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) \geq v_k^\varepsilon(x_0, t_0);$$

and so

$$u_k^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) \leq u_k^\varepsilon(x_0, t_0).$$

Thus

$$\liminf_{\varepsilon \rightarrow 0} u_k^\varepsilon(x_0, t_0) \geq \delta > 0. \quad \square$$

4. Appendix: Identification of the action function

We outline in this section a proof that the functions v_* and v^* introduced in §3 agree and equal I , the unique viscosity solution of

$$(4.1) \quad \begin{cases} \min(I_t + H(DI), I) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ I = 0 & \text{on } G_0 \times \{0\} \\ I = +\infty & \text{on } (\mathbb{R}^n - \bar{G}_0) \times \{0\} \end{cases}$$

First we show

$$(4.2) \quad v_* \geq I \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

For this choose any constant $\mu > 0$ and any function $\zeta \in C^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} \zeta \equiv 0 \text{ on } \bar{G}_0, \zeta > 0 \text{ on } \mathbb{R}^n - \bar{G}_0 \\ 0 \leq \zeta \leq 1. \end{cases}$$

Consider now the auxiliary problem

$$(4.3) \quad \begin{cases} \min(I_{\mu t} + H(DI_{\mu}), I_{\mu}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ I_{\mu} = \mu\zeta & \text{on } \mathbb{R}^n \times (0, \infty), \end{cases}$$

which has a unique, continuous solution according to [6]. Furthermore, according to assertion (ii) in Proposition 3.1 we have

$$(4.4) \quad v_* \geq I_{\mu} \quad \text{on } \mathbb{R}^n \times \{0\}.$$

Finally, v_* is lower semicontinuous, and

$$\min(v_{*t} + H(Dv_*), v_*) \geq 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

in the viscosity sense. Consequently a comparison argument following [3] and [6] implies

$$v_* \geq I_{\mu} \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Letting μ tend to infinity we have

$$I_{\mu} \rightarrow I \quad \text{in } \mathbb{R}^n \times (0, \infty);$$

and so (4.2) follows.

Next we assert that

$$(4.5) \quad v^* \leq I \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

To prove this define for each small $\delta > 0$ the smooth set

$$G_\delta = \{x_0 \in G_0 \mid \text{dist}(x, \mathbb{R}^n - G_0) > \delta\}.$$

Fix $\delta > 0$ and write

$$(4.6) \quad \Lambda_\sigma = \sup_{x \in G_\delta} v^*(x, \sigma) \quad (\sigma > 0).$$

Since v^* is upper semicontinuous and

$$v^* = 0 \quad \text{on } G_0 \times \{0\},$$

according to assertion (ii) of Proposition 3.2, we see that

$$(4.7) \quad \lim_{\sigma \rightarrow 0} \Lambda_\sigma = 0$$

for each fixed $\delta > 0$. Choose some small $\sigma > 0$ and consider the problem

$$(4.8) \quad \begin{cases} \min \left(I_t^{\delta, \sigma} + H(DI^{\delta, \sigma}), I^{\delta, \sigma} \right) = 0 & \text{in } \mathbb{R}^n \times (\sigma, \infty) \\ I^{\delta, \sigma} = \begin{cases} \Lambda_\delta & \text{on } G_\delta \times \{\sigma\} \\ +\infty & \text{on } (\mathbb{R}^n - \bar{G}_\delta) \times \{\sigma\} \end{cases} \end{cases}$$

In view of Proposition 3.2 we have

$$v^* \leq I^{\delta, \sigma} \quad \text{on } \mathbb{R}^n \times \{\sigma\}$$

Since additionally v^* is upper semicontinuous and

$$\min(v_t^* + H(Dv^*), v^*) \leq 0 \quad \text{in } \mathbb{R}^n \times (\sigma, \infty)$$

in the viscosity sense, we have

$$v^* \leq I^{\delta, \sigma} \quad \text{in } \mathbb{R}^n \times (\sigma, \infty).$$

Let $\sigma \rightarrow 0$ and recall (4.7) to discover

$$(4.9) \quad v^* \leq I^\delta \quad \text{on } \mathbb{R}^n \times (0, \infty),$$

when I^δ is the unique viscosity solution of

$$\begin{cases} \min \left[I_t^\delta + H(DI^\delta), I^\delta \right] = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ I^\delta = \begin{cases} 0 \\ +\infty \end{cases} & \begin{array}{l} \text{on } G_\delta \times \{0\} \\ \text{on } (\mathbb{R}^n - \bar{G}_\delta) \times \{0\} \end{array} \end{cases}$$

Now at last send $\delta \rightarrow 0$: since

$$I^\delta \rightarrow I \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

we arrive at (4.5).

Combining (4.2) and (4.5) we have

$$v^* \leq I \leq v_* \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

But since the definitions imply obviously that

$$v_* \leq v^* \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

we have

$$v^* = v_* = I \quad \text{in } \mathbb{R}^n \times (0, \infty). \quad \square$$

Another approach to obtain the above is to modify (in a more or less straight forward way) the results of M.G. Crandall, P.-L. Lions and P.E. Souganidis [4] concerning maximal solutions.

References

1. G. Barles and B. Perthame, Exit time problems in optimal control and vanishing viscosity method, to appear.
2. M.G. Crandall, L.C. Evans, and P.L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations, Trans. AMS 282(1984), 487-502.
3. M.G. Crandall and P.L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Trans. AMS 277 (1983), 1-42.
4. M.G. Crandall, P.L. Lions, and P.E. Souganidis. Universal bounds and maximal solutions for certain evolution equations, to appear in Arch. Anal. Rat. Mech.
5. L.C. Evans and H. Ishii, A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities, Ann. L'Institut H. Poincare 2(1985), 1-20.
6. L.C. Evans and P.E. Souganidis, A PDE approach to geometric optics for certain semilinear parabolic equations, to appear in Indiana University Math. J.
7. L.C. Evans and P.E. Souganidis, A PDE approach to certain large deviation problems for systems of parabolic equations, to appear in Ann. L'Institut H. Poincare.
8. W.H. Fleming, Logarithmic transformations and stochastic control, Advances in Filtering and Optimal Stochastic Control, ed. by W.H. Fleming and L.G. Gorostiza, Springer-Verlag.
9. M.I. Freidlin, Functional Integration and Partial Differential Equations, Annals of Math. Studies 109, Princeton University Press, Princeton, 1985.
10. M.I. Freidlin, Limit theorems for large deviations and reaction-diffusion equations, Annals of Prob. 13(1985), 639-675.
11. H. Ishii, A boundary value problem of Dirichlet type for Hamilton-Jacobi equations.